

HEISENBERG ACTION IN SKEIN THEORY AND EXTERNAL EDGE CONDITION

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ABSTRACT. In this article we give an explicit description of the representation matrix of a Heisenberg type action constructed by Blanchet, Habegger, Masbaum and Vogel. We give the matrix in terms of a ribbon graph and its admissible colorings. We show that components of the representation matrix satisfies the *external edge condition*, which is a natural combinatorial/geometric condition for maps from the first homology of the graph. We give the explicit formula of the trace of the action in the case of surfaces with colored structure using the external edge condition, the Verlinde formula and elementary counting arguments. Our formula is a generalization of the results for a surface without colored structure, which are already known.

1. INTRODUCTION

In [5], Blanchet, Habegger, Masbaum and Vogel constructed a family of topological quantum field theories (TQFT) parameterized by non-negative integers p as a functor from a two dimensional cobordism category ¹ to the category of modules. They constructed a TQFT-module $V_p(C) = V_p(C, l, j')$ for each *surface with colored structure* (C, l, j') . A surface with colored structure is a triple (C, l, j') consisting of a closed oriented surface C , a banded link $l = l_1 \cup \cdots \cup l_n$ in C (i.e. embedded disjoint intervals) and a set of fixed colorings $j' = (j'_1, \cdots, j'_n)$ of components l_1, \cdots, l_n . Here a coloring is an integer. On the other hand, for each *admissible coloring* of a ribbon graph (Definition 2.1), one can associate an element of the TQFT-module. They also showed that such elements form a free basis of their TQFT-module. Our interest in this article is the Heisenberg type action on the TQFT-module for $p = 2k + 4$. They defined involutions on the TQFT-module $V_{2k+4}(C)$ associated with simple closed curves on the surface, and these involutions form a natural action of a Heisenberg type group defined as a central extension of $H_1(C - l; \mathbb{Z}/2)$. Such a Heisenberg action is a starting point of a refinement of their TQFT, which is called *spin-refined TQFT* ([3, 4]).

The purpose of this article is to have an explicit representation matrix of the Heisenberg action on the TQFT-module in terms of a given ribbon graph and its k -admissible colorings based on combinatorial part of Yoshida's construction in [8], and we carry out the computation of the trace of the action for a surface with colored structure using such an explicit description. The main ingredient for the computation of the trace is the *external edge condition*, which is a quite natural geometric/combinatorial condition for a map from the homology of the graph to the coefficient ring of the TQFT-module. The explicit description of the representation matrix tells us that the map appeared in the matrix satisfies the external edge condition. As a corollary of the computation of the trace we have the dimension formula for the *brick decomposition* of the TQFT-module for a surface with colored structure. The formula for a surface without banded link are obtained in [2] and [5] using algebro-geometric and skein theoretical technique respectively. Our explicit computation works uniformly for a surface with non-empty link. The computation of the trace also tells

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¹The domain category considered in [5] is really that of surfaces with p_1 -structure. In this article we do not need any p_1 -structure and we omit it for simplicity.

us that the external edge condition is a characterization of the Heisenberg action on the TQFT-module (Remark 6.1).

We should note that Masbaum gave the computation of the representation matrix a long time ago but he never published it. Such a computation is a special case of the algorithm called *TQFT* that implemented by Masbaum and A'Campo in PARI program to compute representation matrices of the mapping class group on the TQFT-module. It is possible to compute representation matrices of the mapping class group using similar methods. See [1] for the program TQFT and its applications.

This article is organized as follows. In Section 2 we recall some results obtained in [7], which we use in Section 3 to compute the representation matrix explicitly. In Section 3 we first recall the definition of the involutions on the TQFT-modules. We give explicit descriptions of involutions on the TQFT-module associated with a meridian cycle and a longitude cycle. In Subsection 3.1 we introduce the external edge condition (Definition 3.5). By making use of the explicit description one can check that the map appeared in the matrix satisfies the external edge condition (Proposition 3.4). In Section 4 we define four lattices associated with the graph and realize the first homology group of the surface in terms of these lattices (Proposition 4.2). These lattices are introduced in [8] for a closed surface with a pants decomposition. In Section 5 we first recall the definition of the Heisenberg type group in [5], and the explicit description obtained in Section 3 enable us to describe the action of the Heisenberg type group explicitly in terms of the graph and its k -admissible colorings. In Section 6 we compute the trace of the involution for a surface with colored structure (Theorem 6.1). To show the formula we use the external edge condition, the Verlinde formula and elementary counting arguments. In Section 7 we demonstrate the computation of the dimension formulas for a brick decomposition of the TQFT-module for a surface with colored structure.

Convention. In the figures, we use the convention that a line represents a ribbon parallel to the plane and each trivalent vertex is ordered in the counterclockwise direction.

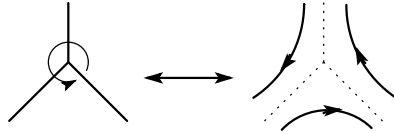


FIGURE 1. Convention of orientation

2. SOME RESULTS IN THE TEMPERLEY-LIEB ALGEBRA

In this section we recall some results obtained in [7], which are used in the next section to obtain the explicit description of involutions on the TQFT-module. Fix a non-negative integer k . We first recall the definition of k -admissible colorings of a graph.

Definition 2.1. Let Γ be a trivalent graph. A map j from the set of edges of Γ to the set of integers $\{0, 1, \dots, k\}$ is called a k -admissible coloring of Γ if the following three conditions hold at each vertex with edges f_1 , f_2 and f_3 ;

$$\begin{cases} j(f_1) + j(f_2) + j(f_3) \in 2\mathbb{Z} \\ |j(f_1) - j(f_2)| \leq j(f_3) \leq j(f_1) + j(f_2) \\ j(f_1) + j(f_2) + j(f_3) \leq 2k. \end{cases}$$

For a vertex with two edges f_1 and $f_2 = f_3$, we interpret these conditions as a corresponding condition with $j(f_2) = j(f_3)$. These three conditions are called the *quantum Clebsh-Gordon condition of level k* .

For a univalent graph Γ with n univalent vertices, we fix an n -tuple of colorings $j' = (j'_1, \dots, j'_n)$ (boundary coloring). A coloring j of Γ is k -admissible if it satisfies the quantum Clebsh-Gordon condition

of level k at each trivalent vertex and it is compatible with j' , i.e, $j(f_l) = j'_l$ for all f_l with a univalent vertex.

The *Jones-Kauffman skein module* of a compact 3-dimensional manifold M over a commutative ring R is defined as the R -module generated by the set of isotopy classes of banded links in M meeting ∂M transversally in the banded link in ∂M quotiented by the Kauffman relations. A colored ribbon graph with a k -admissible coloring in a compact 3-dimensional manifold gives rise to an element in the Jones-Kauffman skein module by expanding the graph at each vertex and inserting the Jones-Wenzl idempotent at each edge. See [5] for detail. The coefficient ring of TQFT-module in [5] is defined as a kind of cyclotomic ring of degree $2(2k+4)$. It contains an indeterminate A which should be a $2(2k+4)$ -th root of unity. Recall the following notations.

- $[n] := \frac{A^{2n} - A^{-2n}}{A^2 - A^{-2}}.$
- $[n]! := [1][2] \cdots [n], ([0]! := 1).$
- $\langle n \rangle := (-1)^n [n+1].$

It is known that $[n]$ is invertible for $n = 1, 2, \dots, k+1$ and $[k+2] = 0$.

Now we recall the results in [7], which we will use in the next section. Those are equalities in the *Temperley-Lieb algebra*. The Temperley-Lieb algebra is the Jones-Kauffman skein module of the box $D^2 \times [0, 1]$ with the standard link in $D^2 \times \{0\}$ and $D^2 \times \{1\}$. The product is given by the standard product of tangles.

Theorem 1. [Fusion coefficient] *Put*

$$\langle a, b, c \rangle = (-1)^{i+j+k} \frac{[i+j+k+1]![i]![j]![k]!}{[i+j]![j+k]![k+i]},$$

where i, j and k (internal colors) are defined by

$$\begin{aligned} i &= \frac{1}{2}(-a + b + c) \\ j &= \frac{1}{2}(a - b + c) \\ k &= \frac{1}{2}(a + b - c). \end{aligned}$$

Then one has the following two equalities in the Temperley-Lieb algebra;

$$\begin{aligned} \begin{array}{c} | \\ a \end{array} \begin{array}{c} | \\ b \end{array} &= \sum_c \frac{\langle c \rangle}{\langle a, b, c \rangle} \begin{array}{c} a \quad b \\ \diagdown \quad \diagup \\ \quad c \\ \diagup \quad \diagdown \\ a \quad b \end{array} \\ \begin{array}{c} \quad c \\ | \\ a \quad b \\ | \\ c' \end{array} &= \delta_{c,c'} \frac{\langle a, b, c \rangle}{\langle c \rangle} \begin{array}{c} \quad c \\ | \\ \quad \end{array} . \end{aligned}$$

Theorem 2. [Tetrahedron coefficient] *Put*

$$\left\langle \begin{array}{ccc} a & b & c \\ d & e & f \end{array} \right\rangle = \frac{\prod_{i=1}^3 \prod_{j=1}^4 [b_i - a_j]!}{[a]![b]![c]![d]![e]![f]!} \begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & \end{pmatrix},$$

where

$$\begin{pmatrix} a_1 & a_2 & a_3 & a_4 \\ b_1 & b_2 & b_3 & \end{pmatrix} = \sum_{\max(a_j) \leq \zeta \leq \min(b_i)} \frac{(-1)^\zeta [\zeta + 1]!}{\prod_{i=1}^3 [b_i - \zeta]! \prod_{j=1}^4 [\zeta - a_j]!}$$

for

$$\begin{aligned} a_1 &= \frac{1}{2}(a + b + c), & b_1 &= \frac{1}{2}(b + c + e + f) \\ a_2 &= \frac{1}{2}(b + d + f), & b_2 &= \frac{1}{2}(a + b + d + e) \\ a_3 &= \frac{1}{2}(c + d + e), & b_3 &= \frac{1}{2}(a + c + d + f) \\ a_4 &= \frac{1}{2}(a + e + f). \end{aligned}$$

Then one has the following equality in the Temperley-Lieb algebra;

Theorem 3. [Half twist coefficient] *Put*

$$\delta(c; a, b) = (-1)^k A^{ij-k(i+j+k+2)},$$

where i, j and k are as in Theorem 1. Then one has the following equality in the Temperley-Lieb algebra;

3. COMPUTATION OF REPRESENTATION MATRICES OF THE INVOLUTIONS

In this section we give graphical computations for involutions on the TQFT-module and obtain the explicit description of them. The involution in [5] is defined as follows. For each simple closed curve c on a oriented surface C , set $Z(c)_0 := (C \times [0, 1], c_{1/2}(k))$, where $c_{1/2}(k)$ is a knot $c \times \{1/2\}$ in $C \times [0, 1]$ colored by k . Then $Z(c)_0$ is a cobordism from C to itself and it induces an endomorphism on the TQFT-module $V_{2k+4}(C)$. The involution in [5] is defined by $Z(c) := (-1)^k Z(c)_0$.

Now we fix a ribbon univalent graph Γ . Let B_Γ be the product of the interval $[0, 1]$ and the ribbon associated with Γ , which is homeomorphic to a handle body. Let C_Γ be the boundary of B_Γ . Note that C_Γ is a closed oriented surface equipped with the pants decomposition which is dual to the given graph Γ , and C_Γ may contain a banded link corresponding to univalent vertices of Γ . We also have a copy $\Gamma_0 = \{0\} \times \Gamma \subset C_\Gamma$ of Γ .

Fix a boundary coloring j' of Γ . For each k -admissible coloring j of Γ , a triple (B_Γ, Γ, j) defines an element of the TQFT-module $V_{2k+4}(C_\Gamma) = V_{2k+4}(C_\Gamma, j')$. In [5] they showed that the elements

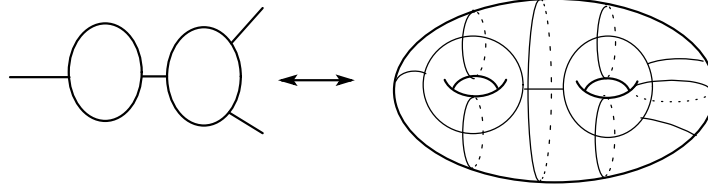


FIGURE 2. Ribbon graph and surface with a pants decomposition

$\{(B_\Gamma, \Gamma, j) \mid j : k\text{-admissible}\}$ form a free basis of $V_{2k+4}(C_\Gamma)$. In this setting the involution explained above is described as follows;

$$Z(c) : (B_\Gamma, \Gamma, j) \mapsto (-1)^k (B_\Gamma \cup (C_\Gamma \times [0, 1]), \Gamma \sqcup c, j \sqcup k),$$

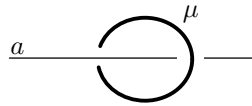
where c is a simple closed curve on C_Γ and j is a k -admissible coloring of Γ . In this section we have an explicit description of this involution for a meridian cycle and a longitude cycle. Here a meridian cycle means an element in the kernel of the natural map $H_1(C_\Gamma) \rightarrow H_1(B_\Gamma)$ and a longitude cycle means an element in $H_1(\Gamma_0)$. Before starting the computation we note the following equalities.

- $[k + 2 - \alpha] = [\alpha]$
- $[k - \alpha]! [\alpha + 1]! = [k + 1]!$
- $\frac{\langle k - a \rangle}{\langle a \ k \ k - a \rangle} = (-1)^a [a + 1] = \langle a \rangle = (-1)^k \langle k - a \rangle$
- $\delta(k - a; k, a) = (-1)^a A^{-a(k+2)}$
- $\frac{\left\langle \begin{matrix} k-A & k-B & C \\ B & A & k \end{matrix} \right\rangle}{\langle k-A \ k-B \ C \rangle} = (-1)^{\frac{A+B-C}{2}} \frac{[k-A]! [k-B]! \left[\frac{A+B+C}{2} + 1 \right]!}{[k+1]! \left[k+1 - \frac{A+B-C}{2} \right]!}$

All these formulas are shown by the direct computations, and the left hand side of last three equalities appear in Theorem 1,2 and 3 for a graph with an edge colored by k . We also note that a triple (k, a, c) is k -admissible if and only if $c = k - a$.

Hereafter we denote an element (B_Γ, Γ, j) by $|j\rangle$ for simplicity.

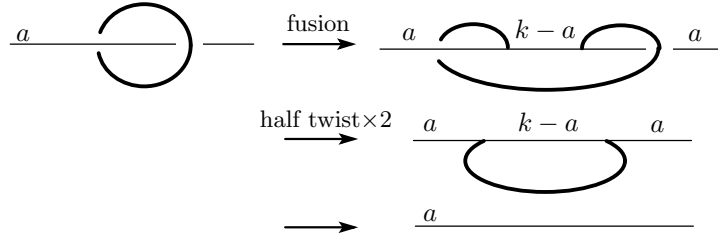
Meridian. Take a meridian circle μ or a circle around a component of the banded link. Fix a k -admissible coloring j of Γ . Assume that μ has an edge colored by a as its core. See the figure below.

FIGURE 3. Meridian μ

Here the thick line represents the cycle colored by k , and we use such a notation throughout this article. By the graphical computation below one has that

$$Z(\mu) : |j\rangle \mapsto (-1)^a |j\rangle.$$

See also Remark 7.6 in [5]. In fact one can compute as follows.



The coefficient associated with these diagrams is

$$\frac{\langle k-a \rangle}{\langle a \ k \ k-a \rangle} \delta(k-a; k, a)^2 \frac{\langle a \ k \ k-a \rangle}{\langle a \rangle} = A^{-2a(k+2)} \frac{\langle k-a \rangle}{\langle a \rangle} = (-1)^{k+a}.$$

Here we used the equality

$$(1) \quad a \text{ --- } \text{loop labeled } k-a \text{ --- } a = \frac{\langle a \ k \ k-a \rangle}{\langle a \rangle} a = \frac{(-1)^k}{\langle a \rangle} a$$

Longitude. Take a cycle λ in $\Gamma_0 \subset C_\Gamma$ and a k -admissible coloring j of Γ . To compute the explicit description of the involution $Z(\lambda)$ on $|j\rangle$ it is enough to consider λ -*external* (resp. *internal*) edges.

Definition 3.1 (λ -external edge and λ -internal edge). For a cycle λ in Γ , an edge $f \in \Gamma$ is said to be a λ -*external edge* if the cycle λ on Γ does not pass through f and one of the vertex of f lies on λ and the other is not. If λ does not pass through f and all vertices of f lie on λ , then we call f is a λ -*internal edge*. See Figure 4. For given λ we denote the set of all λ -external (resp. λ -internal) edges by Ex_λ (resp. In_λ).

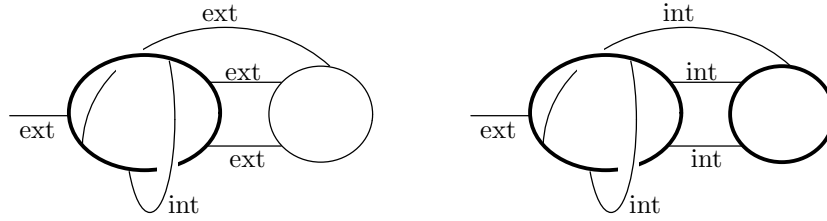


FIGURE 4. External edges and internal edges for λ depicted by thick lines.

By focusing vertices on (a component of) λ and cutting λ -internal edges, one has the colored plane diagram as a part of $\Gamma \sqcup \lambda$ as in Figure 5. The diagram $D(\lambda, j)$ in the box consists of a combination of

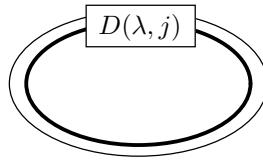


FIGURE 5. Diagram around λ

the two diagrams in Figure 6. (We take a convention for plane diagrams in which we draw the cycle λ over the graph Γ .) Here thick lines in Fig 6 represent a part of $\lambda \subset \Gamma_0$ colored by k and edges colored



FIGURE 6. Diagrams in $D(\lambda, j)$

by c correspond to λ -external/internal edges. We define a number $\epsilon_\lambda(v)$ for each trivalent vertex v in $D(\lambda, j)$ as follows.

$$\epsilon_\lambda(v) = \begin{cases} 0 & (\text{if } v \text{ is type (I)}_{a,b,c}) \\ a - b & (\text{if } v \text{ is type (II)}_{a,b,c}). \end{cases}$$

To describe the involution $Z(\lambda)$ explicitly, let us introduce the action of the homology of the graph on the set of k -admissible colorings.

Definition 3.2. Let Γ be a unitrivalent graph. For a k -admissible coloring j of Γ and a cycle λ on Γ , define $\lambda \cdot j$ by

$$\lambda \cdot j := (\dots, k - j_l, \dots),$$

that is, change all colors on the edges on λ from j_l to $k - j_l$, and the others are the same colors as those in j . One can check that $\lambda \cdot j$ is also a k -admissible coloring and this operation induces an action of $H_1(\Gamma; \mathbb{Z}/2)$ on the set of all k -admissible colorings of Γ .

Proposition 3.3. The involution $Z(\lambda)$ can be described as

$$Z(\lambda) : |j\rangle \mapsto \delta_j(\lambda) |\lambda \cdot j\rangle.$$

Here δ_j is defined by

$$\delta_j(\lambda) := (-1)^{kn_\lambda + \sum j'_l/2} \prod_{f_l \subset \lambda} A^{\epsilon_\lambda(v)(k+2)} \frac{[k - j(f_l)]!}{[j(f_l)]!} \frac{\left[\frac{j(f_l) + j(f_{l+1}) + j'_l}{2} + 1 \right]!}{\left[\frac{k - j(f_l) + k - j(f_{l+1}) + j'_l}{2} + 1 \right]!},$$

where n_λ is the number of edges on λ , v runs vertices on λ and j'_l are colorings on λ -external or internal edges. (See Figure 7.)

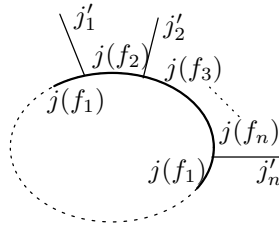


FIGURE 7. Colorings of λ -external/internal edges

Proof. Using formulas explained in the previous section we carry out the graphical computations for the diagrams $(I)_{a,b,c}$ and $(II)_{a,b,c}$.

$$\begin{aligned}
(I)_{a,b,c} ; \quad & \begin{array}{c} c \\ | \\ a \quad b \\ \hline \hline \end{array} \xrightarrow{\text{fusion} \times 2} \begin{array}{c} c \\ | \\ a \quad a \quad b \quad b \\ \hline k-a \quad \quad k-b \end{array} \\
& \xrightarrow{\text{tetrahedron}} \begin{array}{c} c \\ | \\ a \quad \quad b \\ \hline k-a \quad k-b \end{array}
\end{aligned}$$

The coefficient associated with these diagrams is

$$\begin{aligned}
(I)_{a,b,c} ; \quad & \frac{\langle k-a \rangle}{\langle a \ k \ k-a \rangle} \frac{\langle k-b \rangle}{\langle b \ k \ k-b \rangle} \frac{\left\langle \begin{array}{ccc} k-a & k-b & c \\ & b & a \end{array} \right\rangle}{\langle k-a \ k-b \ c \rangle} \\
& = \langle k-a \rangle \langle k-b \rangle \frac{\left\langle \begin{array}{ccc} k-a & k-b & c \\ & b & a \end{array} \right\rangle}{\langle k-a \ k-b \ c \rangle} \\
& = \langle k-a \rangle \langle k-b \rangle (-1)^{\frac{a+b-c}{2}} \frac{[k-a]! [k-b]! \left[\frac{a+b+c}{2} + 1 \right]!}{[k+1]! \left[k+1 - \frac{a+b-c}{2} \right]!}
\end{aligned}$$

$$\begin{aligned}
(II)_{a,b,c} ; \quad & \begin{array}{c} a \quad b \\ \hline | \\ c \end{array} \xrightarrow{\text{fusion} \times 2} \begin{array}{c} a \quad a \quad b \quad b \\ \hline k-a \quad \quad k-b \end{array} \\
& \xrightarrow{\text{fusion}} \begin{array}{c} a \quad a \quad b \quad b \\ \hline k-a \quad \quad k-b \end{array} \\
& \quad \quad \quad \begin{array}{c} c \\ | \\ k-c \end{array} \\
& \quad \quad \quad \begin{array}{c} c \\ | \\ c \end{array} \\
& \xrightarrow{\text{half twist}} \begin{array}{c} a \quad a \quad b \quad b \\ \hline k-a \quad \quad k-b \end{array} \\
& \quad \quad \quad \begin{array}{c} c \\ | \\ k-c \end{array} \\
& \quad \quad \quad \begin{array}{c} c \\ | \\ c \end{array} \\
& \xrightarrow{\text{tetrahedron}} \begin{array}{c} a \quad k-a \quad b \quad b \\ \hline k-c \quad \quad k-b \end{array} \\
& \quad \quad \quad \begin{array}{c} c \\ | \\ c \end{array} \\
& \xrightarrow{\text{tetrahedron}} \begin{array}{c} a \quad k-a \quad k-b \quad b \\ \hline \quad \quad c \end{array}
\end{aligned}$$

The coefficient associated with these diagrams is

$$\begin{aligned}
(\text{II})_{a,b,c} & ; \quad \frac{\langle k-a \rangle}{\langle a \ k \ k-a \rangle} \cdot \frac{\langle k-b \rangle}{\langle b \ k \ k-b \rangle} \cdot \frac{\langle k-c \rangle}{\langle c \ k \ k-c \rangle} \cdot \overline{\delta(k-c; k, c)} \\
& \quad \cdot \frac{\left\langle \begin{array}{ccc} k-a & b & k-c \\ & c & k \end{array} \right\rangle}{\langle k-a \ b \ k-c \rangle} \cdot \frac{\left\langle \begin{array}{ccc} k-a & k-b & c \\ & k & k-c \end{array} \right\rangle}{\langle k-a \ k-b \ c \rangle} \\
& = (-1)^c A^{c(k+2)} \langle k-a \rangle \langle k-b \rangle \langle k-c \rangle \\
& \quad \cdot (-1)^{\frac{a-b+c}{2}} \frac{[k-a]! [k-c]! \left[\frac{a+b+c}{2} + 1\right]!}{[k+1]! \left[k+1 - \frac{a-b+c}{2}\right]!} \cdot (-1)^{\frac{a+b-c}{2}} \frac{[k-b]! [c]! \left[k+1 - \frac{a-b+c}{2}\right]!}{[k+1]! \left[k+1 - \frac{a+b-c}{2}\right]!} \\
& = A^{(2c+c+a-b+c)(k+2)} \langle k-a \rangle \langle k-b \rangle (-1)^{\frac{a+b-c}{2}} \frac{[k-a]! [k-b]! \left[\frac{a+b+c}{2} + 1\right]!}{[k+1]! \left[k+1 - \frac{a+b-c}{2}\right]!} \\
& = A^{(a-b)(k+2)} \times \text{coeff. of } (\text{I})_{a,b,c}.
\end{aligned}$$

Here we note that $A^{2(k+2)} = -1$. By using these computations together with the equality (1) and

$$[k-a]!^2 = [k-a]! \frac{[k+1]!}{[a+1]!},$$

we have the explicit description of $\delta_j(\lambda)$ as in the proposition. \square

Remark 3.1. The computation given here was carried out by Masbaum for a long time ago. Moreover he checked that by rescaling the basis the longitudes act by matrices whose entries are $\pm 1, \pm\sqrt{-1}$ or zero. Namely, put

$$g(j) := \prod_{f: \text{edges of } \Gamma} \frac{1}{[j(f)]!} \prod_{v: \text{vertices of } \Gamma} [j(v) + 1]!$$

for a k -admissible coloring j , where $j(v)$ is defined by

$$j(v) := \frac{j(f_1) + j(f_2) + j(f_3)}{2}$$

for a vertex v with edges f_1, f_2 and f_3 . Then one has

$$Z(\lambda) : g(j)^{-1} |j\rangle \mapsto (-1)^{kn_\lambda + \sum j'_i/2} \prod_{f_i \subset \lambda} A^{\epsilon_\lambda(v)(k+2)} g(\lambda \cdot j)^{-1} |\lambda \cdot j\rangle.$$

3.1. External edge condition. From the description of $Z(\lambda)$ obtained in the previous section, one can see that the trace of $Z(\lambda)$ is equal to the sum of coefficients $\delta_j(\lambda)$ for all k -admissible colorings j with $\lambda \cdot j = j$. For such λ and j , the diagram $D(\lambda, j)$ consists of diagrams $(\text{I})_{k/2, k/2, c}$ and $(\text{II})_{k/2, k/2, c}$. From now on we assume that k is an even number because $k/2$ should be a natural number². We also note that if k is even and a triple $(k/2, k/2, c)$ is admissible then c should be an even number. Then one can check the following proposition from the description of $\delta_j(\lambda)$.

Proposition 3.4. *The family of maps $\delta = (\delta_j)$ appeared in the representation matrices of $Z(\lambda)$ satisfies the external edge condition defined below.*

Definition 3.5 (External edge condition). Let $\alpha = (\alpha_j)$ be a family of maps parameterized by the set of k -admissible colorings of Γ compatible with the given boundary coloring, where each map α_j is a map

²This fact implies that if k is odd then the trace of $Z(\lambda)$ is equal to 0.

from $H_1(\Gamma; \mathbb{Z}/2)$ to the coefficient ring R_k of V_{2k+4} . We say that α satisfies the *external edge condition* if the following condition is satisfied;

$$\alpha_j(\lambda) = (-1)^{\sum_{f_l \in \text{Ex}_\lambda} j_l/2} \quad \text{if } \lambda \cdot j = j.$$

We only note that a λ -internal edge can be thought as a λ -external edge with multiplicity 2, and the coloring on such an edge contributes as $(-1)^{\text{even number}} = 1$.

4. FOUR LATTICES

In this section we realize the first homology group of the surface in terms of a ribbon graph. Let Γ be a univalent ribbon graph with $3g - 3 + 2n$ edges $E(\Gamma) = \{f_l\}$, $2g - 2 + n$ trivalent vertices and n univalent vertices. We decompose the set of edges $E(\Gamma)$ into $E^u(\Gamma) \sqcup E^t(\Gamma)$, where $E^u(\Gamma)$ consists of edges with a univalent vertex and $E^t(\Gamma)$ consists of edges without a univalent vertex. Fix $3g - 3 + 2n$ letters $\check{E}(\Gamma) = \{e_l\}$. These e_l correspond to simple closed curves in the pants decomposition of C_Γ° , the compact surface obtained by removing open disks around the banded link from C_Γ . We also have a decomposition $\check{E}(\Gamma) = \check{E}^u(\Gamma) \sqcup \check{E}^t(\Gamma)$. Here we put $\check{E}^u(\Gamma)$ (resp. $\check{E}^t(\Gamma)$) := $\{e_l \mid f_l \in E^u(\Gamma)$ (resp. $E^t(\Gamma))\}$. Now we introduce four lattices Λ_0 , Λ , Λ_0^* and Λ^* as follows³.

Definition 4.1. Let $\mathbb{R}\langle\check{E}(\Gamma)\rangle$ be the vector space generated by $\check{E}(\Gamma) = \{e_l\}$ over \mathbb{R} . Let $\Lambda_0 := \mathbb{Z}\langle\check{E}(\Gamma)\rangle$ be the standard lattice in $\mathbb{R}\langle\check{E}(\Gamma)\rangle$ and Λ be the lattice generated by the elements $\{E_1^i, E_2^i, E_3^i \mid i = 1, \dots, 2g - 2 + n\}$, where these elements are defined as follows; For each trivalent vertex v_i with edges f_{i_1} , f_{i_2} and f_{i_3} define E_1^i , E_2^i and E_3^i by

$$\begin{aligned} E_1^i &= \frac{1}{2}(-e_{i_1} + e_{i_2} + e_{i_3}) \\ E_2^i &= \frac{1}{2}(e_{i_1} - e_{i_2} + e_{i_3}) \\ E_3^i &= \frac{1}{2}(e_{i_1} + e_{i_2} - e_{i_3}). \end{aligned}$$

If a vertex v_i has only two edges f_{i_1} and $f_{i_2} = f_{i_3}$ then we interpret as

$$\begin{aligned} E_1^i &:= -\frac{1}{2}e_{i_1} + e_{i_2} \\ E_2^i &= E_3^i := \frac{1}{2}e_{i_1}. \end{aligned}$$

Let $\mathbb{R}\langle E^t(\Gamma)\rangle$ be the \mathbb{R} -vector space generated by $E^t(\Gamma)$. Let $\Lambda_0^* = \mathbb{Z}\langle E^t(\Gamma)\rangle$ be the standard lattice in $\mathbb{R}\langle E^t(\Gamma)\rangle$ and Λ^* be the sublattice of Λ_0^* defined by

$$\Lambda^* := \left\{ \sum_l n_l f_l \in \Lambda_0^* \mid n_{i_1} + n_{i_2} + n_{i_3} \in 2\mathbb{Z} \right\}.$$

There is a natural pairing between $\mathbb{R}\langle\check{E}(\Gamma)\rangle$ and $\mathbb{R}\langle E^t(\Gamma)\rangle$ denoted by \cdot , which is defined by $e_l \cdot f_{l'} = \delta_{l,l'}$. Note that this pairing is non-degenerate except the subspace generated by $\check{E}^u(\Gamma)$.

Remark 4.1. Note that Λ is a sublattice of Λ_0 with $\Lambda/\Lambda_0 \cong (\mathbb{Z}/2)^{2g-2+2n-n'}$, where $n' = n$ ($n \geq 1$), 1 ($n = 0$). This follows from the facts $E_1^i \equiv E_2^i \equiv E_3^i \pmod{\Lambda_0}$ for $i = 1, \dots, 2g-2+n$ and $\sum_i E_1^i \equiv 0 \pmod{\Lambda_0}$ if $n = 0$.

Proposition 4.2. *We have the followings;*

- (1) *There is a canonical isomorphism $\Lambda^*/2\Lambda_0^* \cong H_1(\Gamma; \mathbb{Z}/2) (\cong (\mathbb{Z}/2)^g)$.*

³These lattices are introduced in [8] for a closed surface with a pants decomposition.

(2) The inclusion $\Gamma_0 \subset C_\Gamma$ induces a decomposition $H_1(C_\Gamma^\circ; \mathbb{Z}/2) \cong \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^*$.

Proof. 1. The required isomorphism is given as follows. By definition each $\lambda \in \Lambda^*/2\Lambda_0^*$ can be represented by an element in Λ^* which has two (or no) nontrivial entries at each trivalent vertex. Putting edges corresponding to nontrivial entries in λ together we obtain a cycle in Γ . Conversely for a given cycle in Γ we have an element in $\Lambda^*/2\Lambda_0^*$ whose nontrivial entries correspond with edges on the cycle.

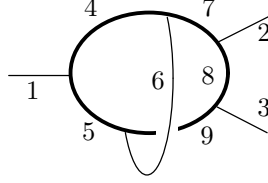


FIGURE 8. The thick line corresponds to $\lambda = f_4 + f_5 + f_7 + f_8 + f_9$

2. The elements in 2Λ correspond to boundaries of the pants decomposition of the surface C_Γ° up to orientation. Hence there is a natural map $\Lambda_0/2\Lambda \rightarrow H_1(C_\Gamma^\circ; \mathbb{Z}/2)$, and one can check that this is an inclusion. Then the rank of cokernel of this inclusion is equal to g because we have that $\Lambda_0/2\Lambda \cong (\mathbb{Z}/2)^{g+n'-1}$. On the other hand we have a natural map $\Lambda^*/2\Lambda_0^* \cong H_1(\Gamma_0; \mathbb{Z}/2) \rightarrow H_1(C_\Gamma^\circ; \mathbb{Z}/2)$ via the inclusion $\Gamma_0 \subset C_\Gamma$. It induces an injection $\Lambda^*/2\Lambda_0^* \rightarrow H_1(C_\Gamma^\circ; \mathbb{Z}/2)/(\Lambda_0/2\Lambda)$, and this is an isomorphism because these have the same rank g . This construction gives a splitting of the exact sequence $0 \rightarrow \Lambda_0/2\Lambda \rightarrow H_1(C_\Gamma^\circ; \mathbb{Z}/2) \rightarrow H_1(C_\Gamma^\circ; \mathbb{Z}/2)/(\Lambda_0/2\Lambda) \cong \Lambda^*/2\Lambda_0^* \rightarrow 0$. \square

Remark 4.2. Note that the subgroup $\Lambda_0^u/2\Lambda^u$ is isomorphic to the subgroup of $H_1(C_\Gamma^\circ; \mathbb{Z}/2)$ generated by boundaries of C_Γ° , where Λ_0^u is the lattice generated by $\check{E}^u(\Gamma)$ and we put $2\Lambda^u := \Lambda_0^u \cap 2\Lambda$.

Remark 4.3. The isomorphism $H_1(C_\Gamma^\circ; \mathbb{Z}/2) \cong \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^*$ is compatible with the intersection pairing. Namely, the mod 2 intersection pairing between meridians and longitudes coincides with the natural pairing between $\Lambda_0/2\Lambda$ and $\Lambda^*/2\Lambda_0^*$ induced from the pairing between Λ_0 and Λ_0^* . We denote this pairing by $\cdot : \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^* \rightarrow \mathbb{Z}/2$, $(\mu, \lambda) \mapsto \mu \cdot \lambda$.

5. REPRESENTATION MATRIX OF THE HEISENBERG ACTION

In this section we describe the Heisenberg action in [5] explicitly in terms of a ribbon graph and its admissible colorings. We first review the construction of the Heisenberg type group in [5]. For an oriented surface C , let $H(C)$ be the direct product of \mathbb{Z} and $H_1(C; \mathbb{Z})$ with multiplication given by $(n, x)(m, y) := (n + m + x \cdot y, x + y)$, where $x \cdot y$ is the intersection number of x and y . We denote $(n, 0)$ by u^n for $n \in \mathbb{Z}$ and $[x] = (0, x)$ for $x \in H_1(C; \mathbb{Z})$.

Definition 5.1. Let $\mathcal{E}(C)$ be the quotient of $H(C)$ by the subgroup generated by u^4 and elements $[2x]$. One can check that $\mathcal{E}(C)$ is a central extension

$$\mathbb{Z}/4 \rightarrow \mathcal{E}(C) \rightarrow H_1(C; \mathbb{Z}/2).$$

For the surface C_Γ associated with a ribbon graph Γ , we denote $\mathcal{E}(C_\Gamma)$ by $\mathcal{E}(\Gamma)$. We can define a map $\tau : \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^* \rightarrow \mathcal{E}(\Gamma)$ as follows. We first note that $\mu \in \Lambda_0/2\Lambda$ is represented by a disjoint union of simple closed curves $\tilde{\mu}$ on C_Γ , and it determined an element in $H_1(C_\Gamma; \mathbb{Z})$ up to sign. The induced element $\tau(\mu) := [\tilde{\mu}] \in \mathcal{E}(\Gamma)$ depends only on μ since $\tilde{\mu} \cdot \tilde{\mu} = 0$ and hence $[\tilde{\mu}] = [-\tilde{\mu}]$. Similarly we have an element $\tau(\lambda) \in \mathcal{E}(\Gamma)$. We define $\tau(\mu, \lambda) := u^{\mu \circ \lambda} \tau(\lambda) \tau(\mu)$, where $\mu \circ \lambda$ is the geometric intersection number of μ and λ .

Remark 5.1. Put $\mathcal{E}'(\Gamma) := \mathbb{Z}/2 \times (\Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^*)$ as a set. Define a group structure on $\mathcal{E}'(\Gamma)$ by

$$(c_1, \mu_1, \lambda_1) \cdot (c_2, \mu, \lambda_2) = (c_1 c_2 (-1)^{\lambda_2 \cdot \mu_1}, \mu_1 + \mu_2, \lambda_1 + \lambda_2),$$

where \cdot is the natural pairing between $\Lambda_0/2\Lambda$ and $\Lambda^*/2\Lambda_0^*$. Then $\mathcal{E}'(\Gamma)$ is a central extension

$$\mathbb{Z}/2 \rightarrow \mathcal{E}'(\Gamma) \rightarrow \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^*.$$

Then one can check that $\mathcal{E}'(\Gamma)$ is a reduction of $\mathcal{E}(\Gamma)$ to a $\mathbb{Z}/2$ -extension. Namely the $\mathbb{Z}/4$ -extension induced from $\mathcal{E}'(\Gamma)$ via the natural inclusion $\mathbb{Z}/2 \hookrightarrow \mathbb{Z}/4$ is isomorphic to $\mathcal{E}(\Gamma)$.

Fix a non-negative integer k and a boundary coloring $j' = (j'_l)$. We have the TQFT-module $V_{2k+4}(C_\Gamma) = V_{2k+4}(C_\Gamma; j')$, which has a basis parameterized by the finite set of all k -admissible colorings. By previous computations, we have the following descriptions of the involutions of meridian cycles and longitude cycles;

Meridian. For $\mu \in \Lambda_0/2\Lambda$ and a k -admissible coloring j one has

$$\tau(\mu) : |j\rangle \mapsto (-1)^{j_\mu} |j\rangle,$$

where j_μ is defined by $j_\mu := \sum_{\epsilon_l \neq 0} j_l$ for $\mu = \sum_l \epsilon_l e_l$.

Longitude. For $\lambda \in \Lambda^*/2\Lambda_0^*$ and a k -admissible coloring j one has

$$\tau(\lambda) : |j\rangle \mapsto \delta_j(\lambda) |\lambda \cdot j\rangle,$$

where the coefficient $\delta_j(\lambda)$ can be computed as in the previous section and it satisfies the *external edge condition* (Definition 3.5).

Combining these descriptions and the action of central elements (See [5, Proposition 7.5].) we have the following description of $\mathcal{E}(\Gamma)$ -action on $V_{2k+4}(C_\Gamma)$;

$$\begin{aligned} \rho^{(k)} : \mathcal{E}(\Gamma) &\rightarrow GL(V_{2k+4}(C_\Gamma)) \\ u^m \tau(\mu, \lambda) &\mapsto \left(|j\rangle \mapsto A^{(k+2)^2(m+\mu \circ \lambda)} (-1)^{(k+1)(m+\mu \circ \lambda) + j_\mu} \delta_j(\lambda) |\lambda \cdot j\rangle \right). \end{aligned}$$

Remark 5.2. If k is an even number the one has that $\rho^{(k)}(\tau(a))\rho^{(k)}(\tau(b)) = (-1)^{\frac{k}{2}a \cdot b} \rho^{(k)}(\tau(a+b))$ by direct computation. In particular in the case of even k , the representation $\rho^{(k)}$ is an abelian.

6. TRACE OF THE HEISENBERG ACTION

In this section we compute the trace of the involutions on the TQFT-module by making use of the previous results. The formula for closed surfaces with empty link is obtained in [2] and [5]. Our computation works uniformly for surfaces with non-empty link, and the formula is the same as the formula for a surface without banded link up to some factor ($\in \{-1, 0, 1\}$) which is determined by the boundary coloring.

Let Γ be a ribbon univalent graph. Fix a boundary coloring j' . For a non-negative integer k we denote by $QCG_k(\Gamma; j')$ the set of all k -admissible coloring of Γ compatible with the given boundary coloring j' . By the description of $\mathcal{E}(\Gamma)$ -action, the trace of $\rho^{(k)}(\tau(\mu, \lambda))$ is given by

$$\text{Tr}(\rho^{(k)}(\tau(\mu, \lambda)); V_{2k+4}(C_\Gamma; j')) = \sum_{j \in QCG_k^\lambda(\Gamma; j')} A^{(k+2)^2 \mu \circ \lambda} (-1)^{(k+1) \mu \circ \lambda + j_\mu} \delta_j(\lambda),$$

where $QCG_k^\lambda(\Gamma; j')$ is the set of all k -admissible colorings fixed by the action of λ :

$$QCG_k^\lambda(\Gamma; j') := \{j \in QCG_k(\Gamma; j') \mid \lambda \cdot j = j\}.$$

In this section we compute the right hand side directly and show the following formula.

Theorem 6.1. *We have the following trace formula;*

$$\text{Tr}(\rho^{(k)}(\tau(\mu, \lambda)); V_{2k+4}(C_\Gamma; j')) = \gamma(j') \frac{1 + (-1)^k}{2} \left(\frac{k+2}{2} \right)^{g-1}$$

for $\mu \in \Lambda_0/2\Lambda$ and $\lambda(\neq 0) \in \Lambda^*/2\Lambda_0^*$ or for $\mu \notin \Lambda_0^u/2\Lambda^u$ and $\lambda = 0$, where $\gamma(j') \in \{-1, 0, 1\}$ is defined below (Definitions/Notations (3)). For $\mu = \sum \epsilon_l e_l \in \Lambda_0^u/2\Lambda^u$, one has $\text{Tr}(\rho^{(k)}(\mu, 0); V_{2k+4}(C_\Gamma; j')) = (-1)^{\sum \epsilon_l j'_l} \# QCG_k(\Gamma; j')$.

The last part follows from the description of the action of a cycle around a component of the banded link. The case for odd k follows from the following lemma.

Lemma 6.2. *If k is an odd number, then we have $\text{Tr}(\rho^{(k)}(\tau(\mu, \lambda)); V_{2k+4}(C_\Gamma; j')) = 0$ for $\mu \in \Lambda_0/2\Lambda$ and $\lambda(\neq 0) \in \Lambda^*/2\Lambda_0^*$ or $\mu \notin \Lambda_0^u/2\Lambda^u$ and $\lambda = 0$.*

Proof. First note that if $\lambda \cdot j = j$ then $j_l = k/2$ for all l with $\lambda_l \neq 0$. But $k/2$ is not an integer if k is odd. This implies that for an odd number k and $\lambda \neq 0$ we have $QCG_k^\lambda(\Gamma; j') = \emptyset$, and hence $\text{Tr}(\rho^{(k)}(\mu, \lambda); V_{2k+4}(C_\Gamma; j')) = 0$. Now we assume that k is an odd number and $\lambda = 0$, $\mu \notin \Lambda_0^u/2\Lambda^u$. In this case we have

$$\text{Tr}(\rho^{(k)}(\mu, 0); V_{2k+4}(C_\Gamma; j')) = \sum_{j \in QCG_k^\lambda(\Gamma; j')} (-1)^{j_\mu} = \# \{j \in QCG_k \mid j_\mu \in 2\mathbb{Z}\} - \# \{j \in QCG_k \mid j_\mu \notin 2\mathbb{Z}\}.$$

Take and fix a cycle $\lambda_\mu \in \Lambda^*/2\Lambda_0^*$ on Γ with $\mu \cdot \lambda_\mu \equiv 1 \pmod{2}$. Then a map

$$\begin{aligned} \{j \in QCG_k \mid j_\mu \in 2\mathbb{Z}\} &\longleftrightarrow \{j \in QCG_k \mid j_\mu \notin 2\mathbb{Z}\} \\ j &\longleftrightarrow \lambda_\mu \cdot j \end{aligned}$$

gives a bijection if k is an odd number because of the relation

$$(\lambda_\mu \cdot j)_\mu = k\lambda_\mu \cdot \mu + j_\mu \pmod{\mathbb{Z}},$$

in particular we obtain $\text{Tr}(\rho^{(k)}(\mu, 0); V_{2k+4}(C_\Gamma; j')) = 0$. \square

Now we make several preparations to show the theorem for even k . Here after we assume that k is an even natural number. We introduce several notations. (See also Example 6.3.)

Definitions/Notations.

- (1) For given $(\mu, \lambda) \in \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^*$ take and fix representatives $\mu = \sum_l \epsilon_l e_l$ and $\lambda = \sum_l \epsilon'_l f_l$ (we denote them by the same letters), and we define $\mu(\lambda)$ by $\mu(\lambda) := \sum_l \epsilon_l (1 - \epsilon'_l) e_l$. We consider the decomposition of the graph Γ into $\Gamma(\lambda) \cup \Gamma'(\lambda)$ and $\Gamma(\lambda) \cup \Gamma'(\lambda)_\mu$ as follows.

$$\Gamma(\lambda) := \{f_l, v_i, w_i \mid f_l \text{ with } \lambda_l \neq 0 \text{ or } f_l \in \text{Ex}_\lambda \cup \text{In}_\lambda, \text{ and } v_i \text{ or } w_i \text{ is a vertex of } f_l\},$$

$$\Gamma'(\lambda) := \{f_l, v_i, w_i \mid f_l \text{ with } \lambda_l = 0 \text{ or } f_l \in \text{Ex}_\lambda, \text{ and } v_i \text{ or } w_i \text{ is a vertex of } f_l\}.$$

$$\Gamma'(\lambda)_\mu; \text{ the graph obtained by cutting the edges in } \Gamma'(\lambda) \text{ with } \mu(\lambda)_l \neq 0.$$

- (2) For a given graph \cdot , we denote by $E^u(\cdot)$ the set of edges which have one univalent vertex and by $E^t(\cdot)$ the set of edges which have no univalent vertex. Note that $E^u(\Gamma(\lambda)) = \text{Ex}_\lambda$. We define the subset of edges as follows;

$$E_\lambda^u := \text{Ex}_\lambda \cap E^u(\Gamma) \subset E^u(\Gamma(\lambda))$$

$$E_\lambda^t := \text{Ex}_\lambda \cap E^t(\Gamma) \subset E^t(\Gamma(\lambda))$$

$$\overline{E}_\lambda^u := E^u(\Gamma) - E_\lambda^u \subset E^u(\Gamma'(\lambda))$$

$$E_\mu^u := E^u(\Gamma'(\lambda)_\mu) - (\text{Ex}_\lambda \cup \overline{E}_\lambda^u) \subset E^u(\Gamma'(\lambda)_\mu).$$

- (3) For a subset E' of $E^u(\cdot)$ for a given graph \cdot , we denote by $j'(E')$ the set of given boundary colorings of E' , and we put $\sum j'(E') := \sum_{f_l \in E'} j'_l$. We also define $\gamma(j'(E')) \in \{-1, 0, 1\}$ by

$$\gamma(j'(E')) := \begin{cases} (-1)^{\sum j'(E')/2} & (\forall f_l \in E', j'_l \in 2\mathbb{Z}) \\ 0 & (\exists f_l \in E', j'_l \notin 2\mathbb{Z}). \end{cases}$$

- (4) We put $n_1 := \#E_\lambda^u$, $n_2 := \#\overline{E}_\lambda^u$ and we assume that $\Gamma(\lambda)$ has $3g_1 - 3 + 2(n_1 + m)$ edges and $\Gamma'(\lambda)$ has $3g_2 - 3 + 2(n_2 + m)$ edges (then $g = g_1 + g_2 + m - 1$ and $m = \#E_\lambda^t$). Note that E_μ^u has an even element, and we put $2m' := \#E_\mu^u$.
- (5) Let $QCG_k^\lambda(\Gamma; j'(\Gamma))$ be the subset of all admissible colorings for $(\Gamma; j'(\Gamma))$ fixed by the action of λ , i.e.,

$$QCG_k^\lambda(\Gamma; j'(\Gamma)) = \{j \in QCG_k(\Gamma; j'(\Gamma)) \mid j_l = k/2 \text{ if } \lambda_l \neq 0\}.$$

Similarly we can define the subset $QCG_k^\lambda(\Gamma(\lambda); j'(\Gamma(\lambda))) \subset QCG_k(\Gamma(\lambda); j'(\Gamma(\lambda)))$.

Example 6.3. An example of $\Gamma(\lambda)$, $\Gamma'(\lambda)$ and $\Gamma'(\lambda)_\mu$ is shown in Figure 9. We take $\mu = e_6 + e_8 + e_{15}$ and $\lambda = f_4 + f_5 + f_7 + f_8 + f_9$. In this case we have $\mu(\lambda) = e_{15}$, $E_\lambda^u = \{f_1\}$, $E_\lambda^t = \{f_{10}, f_{11}\}$, $\overline{E}_\lambda^u = \{f_2, f_3\}$ and $E_\mu^u = \{f_{15}, f_{15}\}$.

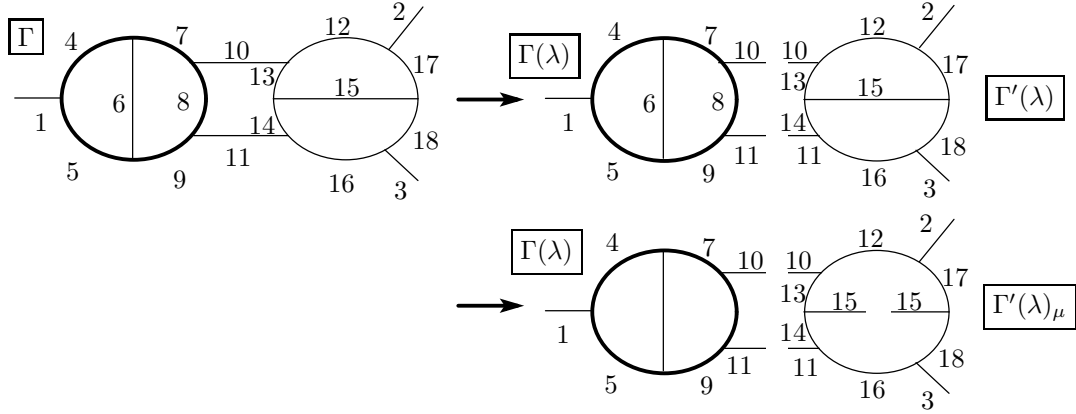


FIGURE 9. Examples of $\Gamma(\lambda)$, $\Gamma'(\lambda)$ and $\Gamma'(\lambda)_\mu$

Theorem 6.1 is shown by the following two propositions.

Proposition 6.4. *We have the equality $\#QCG_k^\lambda(\Gamma(\lambda); j'(\Gamma(\lambda))) = \gamma(j'(Ex_\lambda))^2 \left(\frac{k+2}{2}\right)^{g_1-1}$.*

Proposition 6.5. *We have the equality*

$$\sum_{j'(E_\lambda^t), j'(E_\mu^u)} (-1)^{\sum j'(E_\lambda^t)/2 + \sum j'(E_\mu^u)} \#QCG_k(\Gamma'(\lambda)_\mu; j'(\Gamma'(\lambda)_\mu)) = \gamma(j'(\overline{E}_\lambda^u)) \left(\frac{k+2}{2}\right)^{g_2-1+m}.$$

By using these propositions we can prove Theorem 6.1 by an elementary counting argument.

Proof of Theorem 6.1. Note that for an even number k one has $A^{(k+2)^2\mu\circ\lambda}(-1)^{(k+1)\mu\circ\lambda+j_\mu} = (-1)^{\frac{k}{2}\mu\cdot\lambda+j_\mu}$. By using this equality and the fact that δ_j satisfies the external edge condition, the trace can be computed in the following way;

$$\begin{aligned}
& \sum_{j \in QCG_k^\lambda(\Gamma; j'(\Gamma))} (-1)^{\frac{k}{2}\mu\cdot\lambda+j_\mu} \delta_j(\lambda) = \sum_{j \in QCG_k^\lambda(\Gamma; j'(\Gamma))} (-1)^{\frac{k}{2}\mu\cdot\lambda+j_\mu+\sum_{f_l \in \text{Ex}_\lambda} j_l/2} \\
&= (-1)^{\frac{k}{2}\mu\cdot\lambda} \left(\# \{j \in QCG_k^\lambda \mid j_\mu + \sum_{f_l \in \text{Ex}_\lambda} j_l/2 \equiv 0 \pmod{2}\} - \# \{j \in QCG_k^\lambda \mid j_\mu + \sum_{f_l \in \text{Ex}_\lambda} j_l/2 \equiv 1 \pmod{2}\} \right) \\
&= (-1)^{\frac{k}{2}\mu\cdot\lambda} \left(\# \{j \in QCG_k^\lambda \mid j_{\mu(\lambda)} + \frac{k}{2}\mu\cdot\lambda + \sum_{f_l \in \text{Ex}_\lambda} j_l/2 \equiv 0 \pmod{2}\} \right. \\
&\quad \left. - \# \{j \in QCG_k^\lambda \mid j_{\mu(\lambda)} + \frac{k}{2}\mu\cdot\lambda + \sum_{f_l \in \text{Ex}_\lambda} j_l/2 \equiv 1 \pmod{2}\} \right) \\
&= \# \{j \in QCG_k^\lambda \mid j_{\mu(\lambda)} + \sum_{f_l \in \text{Ex}_\lambda} j_l/2 \equiv 0 \pmod{2}\} - \# \{j \in QCG_k^\lambda \mid j_{\mu(\lambda)} + \sum_{f_l \in \text{Ex}_\lambda} j_l/2 \equiv 1 \pmod{2}\} \\
&= \# \{j \in QCG_k^\lambda \mid j_{\mu(\lambda)}, \sum_{f_l \in \text{Ex}_\lambda} j_l/2 \in 2\mathbb{Z}\} - \# \{j \in QCG_k^\lambda \mid j_{\mu(\lambda)} \in 2\mathbb{Z}, \sum_{f_l \in \text{Ex}_\lambda} j_l/2 \notin 2\mathbb{Z}\} \\
&\quad + \# \{j \in QCG_k^\lambda \mid j_{\mu(\lambda)} \notin 2\mathbb{Z}, \sum_{f_l \in \text{Ex}_\lambda} j_l/2 \notin 2\mathbb{Z}\} - \# \{j \in QCG_k^\lambda \mid j_{\mu(\lambda)} \notin 2\mathbb{Z}, \sum_{f_l \in \text{Ex}_\lambda} j_l/2 \in 2\mathbb{Z}\} \\
&= \sum_{j'(\text{Ex}_\lambda)} (-1)^{\sum j'(\text{Ex}_\lambda)/2} \# QCG_k^\lambda(\Gamma(\lambda); j'(\Gamma(\lambda))) \# \{j \in QCG_k(\Gamma'(\lambda); j'(\Gamma'(\lambda))) \mid j_{\mu(\lambda)} \in 2\mathbb{Z}\} \\
&\quad - \sum_{j'(\text{Ex}_\lambda)} (-1)^{\sum j'(\text{Ex}_\lambda)/2} \# QCG_k^\lambda(\Gamma(\lambda); j'(\Gamma(\lambda))) \# \{j \in QCG_k(\Gamma'(\lambda); j'(\Gamma'(\lambda))) \mid j_{\mu(\lambda)} \notin 2\mathbb{Z}\} \\
&= \left(\frac{k+2}{2} \right)^{g_1-1} \prod_{f_l \in E_\lambda^u} \sin^{2+1} \frac{j'_l+1}{2} \pi \sum_{j'(E_\lambda^t)} (-1)^{\sum j'(E_\lambda^t)/2} \\
&\quad \times \left(\# \{j \in QCG_k(\Gamma'(\lambda); j'(\Gamma'(\lambda))) \mid j_{\mu(\lambda)} \in 2\mathbb{Z}\} - \# \{j \in QCG_k(\Gamma'(\lambda); j'(\Gamma'(\lambda))) \mid j_{\mu(\lambda)} \notin 2\mathbb{Z}\} \right) \\
&= \left(\frac{k+2}{2} \right)^{g_1-1} \prod_{f_l \in E_\lambda^u} \sin \frac{j'_l+1}{2} \pi \sum_{j'(E_\lambda^t)} (-1)^{\sum j'(E_\lambda^t)/2 + \sum j'(E_\mu^u)} \# QCG_k(\Gamma'(\lambda)_\mu; j'(\Gamma'(\lambda)_\mu)) \\
&= \left(\frac{k+2}{2} \right)^{g_1-1} \prod_{f_l \in E_\lambda^u} \sin \frac{j'_l+1}{2} \pi \cdot \left(\frac{k+2}{2} \right)^{g_2-1+m} \prod_{f_l \in E_\lambda^u} \sin \frac{j'_l+1}{2} \pi = \gamma(j') \left(\frac{k+2}{2} \right)^{g-1}.
\end{aligned}$$

Here we used the factorization property in the 6th equal sign, Proposition 6.4 in the 7th equal sign and Proposition 6.5 in the 8th equal sign. \square

Now we start the proof of Proposition 6.4 and 6.5.

Proof of Proposition 6.4. First note that a triple of integers $(k/2, k/2, j_l)$ satisfies the QCG_k -condition if and only if $j_l \in \{0, 2, \dots, k\} \subset 2\mathbb{Z}$, and hence one has that $QCG_k^\lambda(\Gamma(\lambda); j'(\Gamma(\lambda))) \neq \emptyset$ only if $j'(\Gamma(\lambda)) \subset \{0, 2, \dots, k\}^{n_1+m} \subset 2\mathbb{Z}^{n_1+m}$. Next we assume that this condition is satisfied. By the construction there are $g_1 - 1$ edges in $\text{In}_\lambda \subset \Gamma(\lambda)$. On the other hand elements in $QCG_k^\lambda(\Gamma(\lambda); j'(\lambda))$ have coloring $k/2$ at

edges on λ and integer colorings $0, \dots, k$ at edges in In_λ . In addition such integer coloring at edges in In_λ can be taken arbitrary. This implies that there are $\left(\frac{k+2}{2}\right)^{g_1-1}$ elements in $QCG_k^\lambda(\Gamma(\lambda); j'(\Gamma(\lambda)))$. \square

To show Proposition 6.5, we use the following celebrated Verlinde formula.

Verlinde formula. Let Γ be a univalent graph and j' be a boundary coloring of $E^u(\Gamma)$. Then the number of k -admissible colorings compatible with j' is given by the following formula.

$$d_{g,n}(k; j'(\Gamma)) := \# QCG_k(\Gamma; j') = \left(\frac{k+2}{2}\right)^{g-1} \sum_{\nu=1}^{k+1} \frac{\prod_{f_l \in E^u(\Gamma)} \left(\sin \frac{j_l+1}{k+2} \nu \pi\right)}{\left(\sin \frac{\nu \pi}{k+2}\right)^{2g-2+n}}.$$

Proof of Proposition 6.5. By using the factorization property and the Verlinde formula, we have the following computation:

$$\begin{aligned} & \sum_{j'(E_\mu^u)} (-1)^{\sum j'(E_\mu^u)} \# QCG_k(\Gamma'(\lambda)_\mu; j'(\Gamma'(\lambda)_\mu)) \\ &= \sum_{j'(E_\mu^u)} (-1)^{\sum j'(E_\mu^u)} \left(\frac{k+2}{2}\right)^{g_2-m'-1} \sum_{l=1}^{k+1} \frac{\prod_{E_\lambda^t \sqcup \overline{E}_\lambda^u} \left(\sin \frac{j_{\nu'}+1}{k+2} l \pi\right) \prod_{E_\mu^u} \left(\sin^2 \frac{j_{\nu'}+1}{k+2} l \pi\right)}{\left(\sin \frac{l \pi}{k+2}\right)^{2g_2-2+n}} \\ &= \left(\frac{k+2}{2}\right)^{g_2-m'-1} \sum_{l=1}^{k+1} \frac{\prod_{E_\lambda^t \sqcup \overline{E}_\lambda^u} \left(\sin \frac{j_{\nu'}+1}{k+2} l \pi\right) \sum_{j'(E_\mu^u)} (-1)^{\sum j'(E_\mu^u)} \prod_{E_\mu^u} \left(\sin^2 \frac{j_{\nu'}+1}{k+2} l \pi\right)}{\left(\sin \frac{l \pi}{k+2}\right)^{2g_2+n-2}} \\ &= \left(\frac{k+2}{2}\right)^{g_2-1} \prod_{E_\lambda^t \sqcup \overline{E}_\lambda^u} \sin \frac{j_{\nu'}+1}{2} \pi. \end{aligned}$$

Here we used the following elementary equalities in the last equal sign;

$$\begin{aligned} & \sum_{j'(E_\mu^u)} (-1)^{\sum j'(E_\mu^u)} \prod_{E_\mu^u} \left(\sin^2 \frac{j_{\nu'}+1}{k+2} l \pi\right) \\ &= \sum_{j'_1, \dots, j'_m} (-1)^{j'_1 + \dots + j'_m} \prod_{\nu'=1}^{m'} \left(\sin^2 \frac{j_{\nu'}+1}{k+2} l \pi\right) = \begin{cases} \left(\frac{k+2}{2}\right)^{m'} & (l = \frac{k+2}{2}) \\ 0 & (l \neq \frac{k+2}{2}). \end{cases} \end{aligned}$$

Then the number in the proposition can be computed as follows.

$$\begin{aligned} & \left(\frac{k+2}{2}\right)^{g_2-1} \sum_{j'(E_\lambda^t)} (-1)^{\sum j'(E_\lambda^t)} \prod_{E_\lambda^t \sqcup \overline{E}_\lambda^u} \sin \frac{j_{\nu'}+1}{2} \pi \\ &= \left(\frac{k+2}{2}\right)^{g_2-1} \prod_{\overline{E}_\lambda^u} \sin \frac{j_{\nu'}+1}{2} \pi \left(\sum_{a=0}^k (-1)^a \sin \frac{a+1}{2} \pi\right)^m = \left(\frac{k+2}{2}\right)^{g_2-1+m} \prod_{\overline{E}_\lambda^u} \sin \frac{j_{\nu'}+1}{2} \pi. \end{aligned}$$

\square

Remark 6.1. We can show that the external edge condition is a characterization of the Heisenberg action on $V_{2k+4}(C_\Gamma)$ in the following sense. Let Γ be a univalent graph and k a non-negative integer. Fix a boundary coloring j' of Γ and let $R_k(\Gamma; j')$ be the free R_k -module generated by the finite set $QCG_k(\Gamma; j')$,

which is isomorphic to $V_{2k+4}(C_\Gamma; j')$. Take a family of maps $\alpha = (\alpha_j : H_1(\Gamma; \mathbb{Z}/2) \rightarrow R_k)$ parameterized by $j \in QCG_k(\Gamma; j')$. We assume that α satisfies the *cocycle condition*

$$\alpha_j(\lambda_1 + \lambda_2) = \alpha_{\lambda_2 \cdot j}(\lambda_1) \alpha_j(\lambda_2) \quad (j \in QCG_k(\Gamma; j'), \lambda_1, \lambda_2 \in H_1(\Gamma; \mathbb{Z}/2)).$$

By using such α we can define a homomorphism

$$\begin{aligned} \rho_\alpha^{(k)} : \mathcal{E}(\Gamma) &\rightarrow GL(R_k(\Gamma; j')) \\ u^m \tau(\mu, \lambda) &\mapsto \left(|j\rangle \mapsto A^{(k+2)^2(m+\mu \circ \lambda)} (-1)^{(k+1)(m+\mu \circ \lambda) + j_\mu} \alpha_j(\lambda) |\lambda \cdot j\rangle \right). \end{aligned}$$

Our proof of Theorem 6.1 shows that if $\alpha = (\alpha_j)$ satisfies the external edge condition, then the representation $\rho_\alpha^{(k)}$ on $R_k(\Gamma; j')$ is equivalent to the representation $\rho^{(k)}$ on $V_{2k+4}(C_\Gamma; j')$. In a suitable categorical setting, we can show that the converse is true. See [6] for detail.

7. BRICK DECOMPOSITION

Our trace formula (Theorem 6.1) enable one to describe the dimension of the direct summands of canonical decomposition of the TQFT-module $V_{2k+4}(C)$ associated with the $\mathcal{E}(C)$ -action. Such a decomposition is called the *brick decomposition* in [2].

Theorem. ([5, Theorem 1.6]) *Let (C, l, c) be a surface with colored structure.*

- (i) *If $k \equiv 0 \pmod{4}$, then the action of $\mathcal{E}(C)$ decomposes $V_{2k+4}(C)$ into a direct sum of subspaces $V_{2k+4}(C, h)$, canonically associated with cohomology classes $h \in H^1(C - l; \mathbb{Z}/2)$.*
- (ii) *If $k \equiv 2 \pmod{4}$, then the action of $\mathcal{E}(C)$ decomposes $V_{2k+4}(C)$ into a direct sum of subspaces $V_{2k+4}(C, q)$, canonically associated with spin structures q on $C - l$.*

These subspaces are defined as follows;

$$V_{2k+4}(C, h) = \{v \in V_{2k+4}(C) \mid \rho^{(k)}(\tau(a))v = h(a)v \text{ for all } a \in H_1(C - l; \mathbb{Z}/2)\}$$

$$V_{2k+4}(C, q) = \{v \in V_{2k+4}(C) \mid \rho^{(k)}(\tau(a))v = q(a)v \text{ for all } a \in H_1(C - l; \mathbb{Z}/2)\}.$$

Here we used identifications $H^1(C - l; \mathbb{Z}/2) = \text{Hom}(H_1(C - l; \mathbb{Z}/2), \mathbb{Z}/2)$ and $\text{Spin}(C - l)$ with the set of quadratic forms on $H_1(C - l; \mathbb{Z}/2)$ inducing the intersection form.

Remark 7.1. They also showed in [2] and [5] that if k is odd, then $V_{2k+4}(C)$ is isomorphic to the tensor product of certain lower level modules as a representation of $\mathcal{E}(C)$.

In [2] and [5] they gave the dimension formula for submodules $V_{2k+4}(C, h)$ and $V_{2k+4}(C, q)$ for a closed surface equipped with the empty link. Theorem 6.1 tells us that the similar formulas hold for a surface with non-empty link. We demonstrate the computation as follows. The computation itself is the almost same as in [2] except the consideration of the boundary.

Now we start from a ribbon unitrivalent graph Γ . Then we have the associated closed surface $C = C_\Gamma$ and the isomorphism $H_1(C^\circ; \mathbb{Z}/2) = \Lambda_0/2\Lambda \oplus \Lambda^*/2\Lambda_0^*$, where C° is the compact surface obtained by removing open discs around the banded link in C .

Remark 7.2. As it is noted in [5], the action of $\mu = \sum \epsilon_l e_l \in \Lambda_0^u/2\Lambda^u$ represented by boundary circles is given by scalar multiplication of $(-1)^{\sum \epsilon_l j_l^i}$, and hence $V_{2k+4}(C, h)$ (resp. $V_{2k+4}(C, q)$) is zero except if $h(e_l) = (-1)^{j_l^i}$ (resp. $q(e_l) = (-1)^{j_l^i}$) for all boundary circles e_l . We denote by $H^1(C; j')$ (resp. $\text{Spin}(C; j')$) the set of cohomology classes (resp. spin structures) on $C - l$ satisfying $h(e_l) = (-1)^{j_l^i}$ (resp. $q(e_l) = (-1)^{j_l^i}$) for all $e_l \in \Lambda_0^u/2\Lambda^u$.

The case $k \equiv 0 \pmod{4}$.

We have a direct sum decomposition

$$V_{2k+4}(C) = V_{2k+4}(C; j') = \bigoplus_h V_{2k+4}(C, h; j').$$

The trace formula asserts that the trace takes the same value for all nontrivial non-boundary element in $H_1(C^\circ; \mathbb{Z}/2)$, and is therefore invariant under the automorphism group of $H_1(C^\circ; \mathbb{Z}/2)$. One can check that the action of automorphism group on $H^1(C; j')$ is transitive. These imply that the dimension of $V_{2k+4}(C : h)$ is same for all non-trivial $h \in H^1(C; j')$. We denote this dimension by $d_{g,n}^{(1)}(k; j')$, and we put $d_{g,n}^{(0)}(k; j') := \dim V_{2k+4}(C, 0; j')$. These numbers can be computed by the following two equalities;

$$\begin{aligned} d_{g,n}(k; j') &= d_{g,n}^{(0)}(k; j') + (2^{2g} - 1)d_{g,n}^{(1)}(k; j') \\ \gamma(j') \left(\frac{k+2}{2} \right)^{g-1} &= \text{Tr}(\tau_a) = d_{g,n}^{(0)}(k; j') - d_{g,n}^{(1)}(k; j') \quad (a \notin \Lambda_0^u / 2\Lambda^u). \end{aligned}$$

Note that $\#H^1(C; j') = 2^{2g}$ and $d_{g,n}(k; j') = \dim V_{2k+4}(C; j')$ is the Verlinde number. In this way for $k \equiv 0 \pmod{4}$, one has that

$$\begin{aligned} d_{g,n}^{(0)}(k; j') &= \frac{1}{2^{2g}} \left(d_{g,n}(k; j') + (2^{2g} - 1)\gamma(j') \left(\frac{k+2}{2} \right)^{g-1} \right) \\ d_{g,n}^{(1)}(k; j') &= \frac{1}{2^{2g}} \left(d_{g,n}(k; j') - \gamma(j') \left(\frac{k+2}{2} \right)^{g-1} \right). \end{aligned}$$

The case of $k \equiv 2 \pmod{4}$.

We have a direct sum decomposition

$$V_{2k+4}(C) = V_{2k+4}(C; j') = \bigoplus_q V_{2k+4}(C, q; j').$$

As in the same way for $k \equiv 0 \pmod{4}$, the trace formula asserts that the trace is invariant under the automorphism group of $H_1(C^\circ; \mathbb{Z}/2)$. Fix a decomposition $H_1(C^\circ; \mathbb{Z}/2) = \Lambda_0^u / 2\Lambda^u \oplus (\Lambda_0^u / 2\Lambda^u)^\perp$, $(\Lambda_0^u / 2\Lambda^u)^\perp := H_1(C^\circ; \mathbb{Z}/2) / (\Lambda_0^u / 2\Lambda^u)$. According to this decomposition we have the decomposition $q(a) = q_1(a_1)q_2(a_2)$, for $a = (a_1, a_2) \in H_1(C^\circ; \mathbb{Z}/2)$. Since it is enough to consider for elements in $\text{Spin}(C; j')$, we may assume that the map q_1 is equal to the map $\sum \varepsilon_l e_l \mapsto (-1)^{\sum \varepsilon_l j_l}$. On the other hand there is a well-defined *Arf invariant* $\text{Arf}(q_2) \in \mathbb{Z}/2$, which parameterizes isomorphism classes of quadratic forms. In this way one knows that the dimension of $V_{2k+4}(C, q; j')$ depends only on $\text{Arf}(q) := \text{Arf}(q_2)$. For $\varepsilon = +$ or $-$, we put $d_{g,n}^\varepsilon(k; j') := \dim V_{2k+4}(C, q_\varepsilon; j')$, where q_ε has Arf invariant $\varepsilon \in \{\pm 1\}$. These numbers can be computed by the following two equalities;

$$\begin{aligned} d_{g,n}(k; j') &= 2^{g-1}(2^g + 1)d_{g,n}^+(k; j') + 2^{g-1}(2^g - 1)d_{g,n}^-(k; j') \\ \gamma(j') \left(\frac{k+2}{2} \right)^{g-1} &= \text{Tr}(\tau(a)) = 2^{g-1}(d_{g,n}^+(k; j') - d_{g,n}^-(k; j')) \quad (a \notin \Lambda_0^u / 2\Lambda^u). \end{aligned}$$

Here we used the facts that for $a \notin \Lambda_0^u / 2\Lambda^u$ and $\varepsilon = +, -$, one has

- $\# \{q_2 : (\Lambda_0^u / 2\Lambda^u)^\perp \rightarrow \mathbb{Z}/2 \mid \text{Arf}(q_2) = 0\} = 2^{g-1}(2^g - 1)$
- $\# \{q_2 : (\Lambda_0^u / 2\Lambda^u)^\perp \rightarrow \mathbb{Z}/2 \mid q_2(a) = -1, \text{Arf}(q_2) = \varepsilon\} = 2^{2g-2}$.

In this way for $k \equiv 2 \pmod{4}$, one has that

$$d_{g,n}^\varepsilon(k; j') = \frac{1}{2^{2g}} \left(d_{g,n}(k; j') + (\varepsilon 2^g - 1) \gamma(j') \left(\frac{k+2}{2} \right)^{g-1} \right).$$

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REFERENCES

- [1] N. A'Campo, *TQFT computations and experiments*, arXiv:math.GT/0312016.
- [2] J. E. Andersen and G. Masbaum, *Involutions on moduli spaces and refinements of the Verlinde formula*, Math. Ann., 314 (1999), 291-326.
- [3] C. Blanchet, *A spin decomposition of the Verlinde formulas for type A modular categories*, Comm. Math. Phys., 257 (2005), No. 1, 1-28.
- [4] C. Blanchet and G. Masbaum, *Topological quantum field theories for surfaces with spin structure*, Duke Math. J., 82 (1996), 229-267.
- [5] C. Blanchet, N. Habegger, G. Masbaum and P. Vogel, *Topological quantum field theories derived from the Kauffman bracket*, Topology 34, No.4 (1995), 883-927.
- [6] H. Fujita, *External edge condition and group cohomologies associated with the quantum Clebsch-Gordan condition*, arXiv:math.GT/0708.4190, to appear in Kyushu J. Math..
- [7] G. Masbaum and P. Vogel, *3-valent graphs and the Kauffman bracket*, Pacific J.Math. 164 (1994), 361-381.
- [8] T. Yoshida *An abelianization of $SU(2)$ Wess-Zumino-Witten model*, Ann. of Math., 164, No.1 (2006), 1-49.

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